Dynamical *r*-matrix for the elliptic Ruijsenaars - Schneider system

1996 J. Phys. A: Math. Gen. 29 L333

(http://iopscience.iop.org/0305-4470/29/13/005)

J. Phys. A: Math. Gen. **29** (1996) L333–L340. Printed in the UK

LETTER TO THE EDITOR

Dynamical *r*-matrix for the elliptic Ruijsenaars–Schneider system

F W Nijhoff†, V B Kuznetsov†‖, E K Sklyanin‡¶ and O Ragnisco§

† Department of Applied Mathematical Studies, University of Leeds, Leeds LS2 9JT, UK
‡ Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606, Japan
§ Dipartimento di Fisica, Terza Università di Roma, Via Vasca Navale 84, Rome, Italy

Received 25 March 1996, in final form 2 May 1996

Abstract. The classical *r*-matrix structure for the generic elliptic Ruijsenaars–Schneider model is presented. It makes manifest the integrability of this model as well as of its discrete-time version that was constructed in a recent paper.

The problem of finding a classical *r*-matrix structure for the Calogero–Moser (CM) type of models aroused some attention a few years ago, see [1–3]. The fact that this had remained an open problem until relatively recently probably lies in the specific feature that for these models the *r*-matrix turns out to be of a dynamical type, i.e. it depends on the dynamical variables. Similar features have been found in other integrable many-body problems as well, e.g. systems separable in the generalized ellipsoidal coordinates [4]. The difficulty presented by the dynamical aspect of the *r*-matrix is that the Poisson algebra of a model whose structural constants are given by a dynamical *r*-matrix is, generally speaking, no longer closed, and that there is no closed-form Yang–Baxter equation defining the *r*-matrix. So far, only for one particular example—the spin generalization of the Calogero–Moser model—a proper algebraic setting (the Gervais–Neveu–Felder equation) is found [5] which also allows one to quantize the model. For other models finding the algebraic interpretation of the dynamical *r*-matrix and solving the quantization problem are still open questions.

One of the most important integrable many-body systems is the relativistic variant of the Calogero–Moser model, the so-called Ruijsenaars–Schneider (RS) model introduced in [6, 7]. Its importance lies in the fact that it can be considered as a q-deformation of the CM model and as such the corresponding quantum model is realized in terms of commuting difference operators whose eigenfunctions are given in terms of Macdonald polynomials, see, e.g., [8, 9]. On the classical level, a dynamical *r*-matrix was found only very recently in [10] for the rational and trigonometric (hyperbolic) cases, although a special parameter-case was already treated in an earlier paper, [11]. A geometric interpretation was given in a recent preprint, see [12]. So far, no results have been found for the full elliptic case. That, in fact, is the subject of this letter where we will present the dynamical *r*-matrix structure for the RS model in the generic elliptic case, thus generalizing the previous results of [10–12].

‖ On leave from: Department of Mathematical and Computational Physics, Institute of Physics, St Petersburg University, St Petersburg 198904, Russia.
¶ On leave from: Steklov Mathematical Institute, Fontanka 27, St Petersburg 191011, Russia.

0305-4470/96/130333+08$19.50

Let us describe the Ruijsenaars–Schneider model and its discrete-time version. The equations of motion of the RS model in its generic (elliptic) form read

$$\ddot{q}_i = \sum_{j \neq i} \dot{q}_i \dot{q}_j v(q_i - q_j) \qquad i = 1, \ldots, N \tag{1a}$$

where the potential $v(x)$ is given by

$$v(x) = \frac{\wp'(x)}{\wp(\lambda) - \wp(x)} \tag{1b}$$

in which $\wp(x) = \wp(x|\omega_1, \omega_2)$ is the Weierstrass P-function, $2\omega_{1,2}$ being a pair of periods, and λ is the (relativistic) deformation parameter. As shown by Ruijsenaars and Schneider in [6, 7], this multi-particle model is integrable, and carries a representation of the Poincaré algebra in two dimensions. Moreover, a large number of the characteristics of the CM model are generalized in a natural way to the relativistic case, such as the existence of a Lax pair, a sufficient number of integrals of the motion in involution, and exact solution schemes in the special cases of rational and trigonometric/hyperbolic limits. The elliptic case has recently been investigated by Krichever and Zabrodin in [13] in connection with the non-Abelian Toda chain.

In [14] an exact time-discretization of equations (1a) was constructed, given by an integrable correspondence of the form

$$\prod_{\substack{k=1\\k\neq\ell}}^{N} \frac{\sigma(q_\ell - q_k + \lambda)}{\sigma(q_\ell - q_k - \lambda)} = \prod_{k=1}^{N} \frac{\sigma(q_\ell - \widetilde{q}_k)\, \sigma(q_\ell - \underset{\sim}{q}_k + \lambda)}{\sigma(q_\ell - \underset{\sim}{q}_k)\, \sigma(q_\ell - \widetilde{q}_k - \lambda)} \qquad \ell = 1, \ldots, N\,. \tag{2}$$

In (2) the q_k denote the particle positions for the time variable equal to n, the tilde being a shorthand notation for the discrete-time shift, i.e. for $q_k(n) = q_k$ we write $q_k(n+1) = \widetilde{q}_k$, and $q_k(n-1) = \underset{\sim}{q}_k$. The function $\sigma(x)$ is the Weierstrass sigma-function, (see the appendix for the definition), and λ is the parameter of the system as in the continuous case (1).

The initial value problem for equations (2), given initial particle positions $\{q_i(0)\}$ and $\{q_i(1)\}$, leads to the problem of solving, at each iteration step, a coupled system of N algebraic equations for N unknowns, and it was shown in [14] that in fact it is an integrable symplectic correspondence (for a definition, see e.g. [15]) with respect to the standard symplectic form $\Omega = \sum_k \mathrm{d}p_k \wedge \mathrm{d}q_k$. This implies that any branch of the correspondence given by equations (2) defines a canonical transformation with respect to the standard Poisson brackets given by

$$\{p_k, q_\ell\} = \delta_{k\ell} \qquad \{p_k, p_\ell\} = \{q_k, q_\ell\} = 0\,. \tag{3}$$

Here

$$p_\ell = \sum_{k=1}^{N} \left(-\log|\sigma(q_\ell - \underset{\sim}{q}_k)| + \log|\sigma(q_\ell - \underset{\sim}{q}_k + \lambda)| \right)\,. \tag{4}$$

The discrete equations of motion (2) arise from a discrete Lax pair of the form

$$L_\kappa = \sum_{i,j=1}^{N} h_i^2 \Phi_\kappa(q_i - q_j + \lambda) e_{ij} \tag{5a}$$

$$M_\kappa = \sum_{i,j} \widetilde{h}_i^2 \Phi_\kappa(\widetilde{q}_i - q_j + \lambda) e_{ij} \tag{5b}$$

using the discrete Lax equation

$$\widetilde{L}_\kappa M_\kappa = M_\kappa L_\kappa\,. \tag{6}$$

Notice here that in (5) we use a different gauge from the symmetric one used in [14]. In equations (5) the variable κ is an additional spectral parameter, and the matrices e_{ij} are the standard elementary matrices whose entries are given by $(e_{ij})_{k\ell} = \delta_{ik}\delta_{j\ell}$. The function Φ_κ is called the Baker function and is defined as

$$\Phi_\kappa(x) \equiv \frac{\sigma(x+\kappa)}{\sigma(x)\sigma(\kappa)} \tag{7}$$

which obeys a number of functional relations listed in the appendix. The auxiliary variables h_ℓ^2 can be expressed in terms of the canonical variables, we obtain

$$h_\ell^2 = e^{p_\ell} \prod_{k\neq\ell} \frac{\sigma(q_\ell - q_k - \lambda)}{\sigma(q_\ell - q_k)} \,. \tag{8}$$

In terms of these variables we have the following Poisson brackets:

$$\begin{aligned} &\{q_k, q_\ell\} = 0 \qquad \{\log h_k^2\,, q_\ell\} = \delta_{k\ell} \\ &\{\log h_k^2\,,\, \log h_\ell^2\} = \zeta(q_k - q_\ell + \lambda) + \zeta(q_k - q_\ell - \lambda) - 2\zeta(q_k - q_\ell) \qquad k \neq \ell\,. \end{aligned} \tag{9}$$

It is easy to see that in terms of the canonical variables p_ℓ and q_ℓ, the Lax matrix L_κ in (5*a*) is exactly the same as the one of the continuous RS model, see [16]. In fact, taking the continuum limit on the discrete-time part of the Lax pair (5), namely the matrix M_κ (5*b*), we obtain a Lax pair for the continuous RS model given by equations (1*a*). Since the L_κ-matrix for the discrete and continuous models is the same, the proof of involutivity of the invariants (integrals) $I_\ell = \mathrm{tr}\, L_\kappa^\ell$ is the same in both cases, and sufficient to assess the Liouville integrability both discrete as well as continuous. The proof can be found in the original paper of Ruijsenaars [7], but is rather involved. Having at one's disposal an *r*-matrix structure would make the involutivity manifest. So far such an *r*-matrix has not been found in the full elliptic case. We will now proceed to establish this *r*-matrix structure.

As was noted recently by Suris, see [12], the main difference between the *r*-matrix structures of the relativistic and non-relativistic CM models lies in the fact that the latter is given in terms of a linear Lie–Poisson structure/bracket, whereas the former is given in terms of a quadratic bracket, see also [11]. The Poisson structure for the RS model will thus be given in the following quadratic *r*-matrix form (see [17, 18]):

$$\begin{aligned} \{L_\kappa \overset{\otimes}{,} L_{\kappa'}\} = {} & L_\kappa \otimes L_{\kappa'} r^-_{\kappa,\kappa'} - r^+_{\kappa,\kappa'} L_\kappa \otimes L_{\kappa'} + (L_\kappa \otimes \mathbf{1})\, s^+_{\kappa,\kappa'}\, (\mathbf{1} \otimes L_{\kappa'}) \\ & - (\mathbf{1} \otimes L_{\kappa'})\, s^-_{\kappa,\kappa'}\, (L_\kappa \otimes \mathbf{1})\,. \end{aligned} \tag{10}$$

The following symmetry conditions must hold for the *r*-matrices, $r^\pm_{\kappa,\kappa'}$ and $s^\pm_{\kappa,\kappa'}$:

$$P r^\pm_{\kappa,\kappa'} P = -r^\pm_{\kappa',\kappa} \qquad P s^+_{\kappa,\kappa'} P = s^-_{\kappa',\kappa} \tag{11a}$$

where P is the permutation matrix in the tensor product of two matrices, i.e. $P(A \otimes B)P = B \otimes A$, as well as the condition

$$r^+_{\kappa,\kappa'} - s^+_{\kappa,\kappa'} = r^-_{\kappa,\kappa'} - s^-_{\kappa,\kappa'} \tag{11b}$$

in order that the quadratic Poisson algebra generates Hamiltonian flows for the invariants of the model, see [12, 19]. The condition (11*b*) was also formulated in [19] in order to allow for a quadratic algebra on the lattice in terms of a local Lax representation to be integrated to a quadratic algebra in terms of the monodromy matrix.

The choice of a gauge for the Lax matrices seems to be quite important in that it influences to a great extent the complexity of the associated *r*-matrix. The Lax matrix (5*a*) has the nice property that it yields a remarkably simple *r*-matrix structure even in the elliptic case. In fact, the *r*-matrices we found are of the form

$$r^{-}_{\kappa,\kappa'} = r_{\kappa,\kappa'} - s_\kappa + P s_{\kappa'} P \tag{12a}$$

$$r^{+}_{\kappa,\kappa'} = r_{\kappa,\kappa'} + u^{+} + u^{-} \tag{12b}$$

$$s^{+}_{\kappa,\kappa'} = s_\kappa + u^{+} \tag{12c}$$

$$s^{-}_{\kappa,\kappa'} = P s_{\kappa'} P - u^{-} \tag{12d}$$

where†

$$r_{\kappa,\kappa'} = r^{0}_{\kappa,\kappa'} + \sum_i \zeta(\kappa-\kappa')e_{ii}\otimes e_{ii} + \sum_{i\neq j}\zeta(q_i-q_j)e_{ii}\otimes e_{jj} \tag{13a}$$

$$r^{0}_{\kappa,\kappa'} = \sum_{i\neq j}\Phi_{\kappa-\kappa'}(q_i-q_j)e_{ij}\otimes e_{ji} \tag{13b}$$

$$s_\kappa = \sum_{i,j}\left(L_\kappa^{-1}\partial_\lambda L_\kappa\right)_{ij} e_{ij}\otimes e_{jj} \tag{13c}$$

$$u^{\pm} = \sum_{i,j}\zeta(q_j-q_i\pm\lambda)e_{ii}\otimes e_{jj}\,. \tag{13d}$$

The matrix elements in (13*c*) can be calculated explicitly using equation (A11) for the inverse of the elliptic Cauchy matrix as well as making diligent use of the elliptic Lagrange interpolation formulae (A8) and (A9), and this yields the following expression:

$$\begin{aligned}\left(L_\kappa^{-1}\partial_\lambda L_\kappa\right)_{ij} = \delta_{ij}&\left[\zeta(\kappa+N\lambda)-\zeta(\lambda)+\sum_{k\neq i}\left(\zeta(q_i-q_k-\lambda)-\zeta(q_i-q_k)\right)\right]\\ &+(1-\delta_{ij})\left[\prod_{\substack{k=1\\k\neq i}}^{N}\frac{\sigma(q_i-q_k-\lambda)}{\sigma(q_i-q_k)}\right]\left[\prod_{\substack{k=1\\k\neq j}}^{N}\frac{\sigma(q_j-q_k)}{\sigma(q_j-q_k-\lambda)}\right]\Phi_{\kappa+N\lambda}(q_i-q_j)\,.\end{aligned} \tag{14}$$

The proof of the *r*-matrix structure (10) together with (12) and (13) is by direct computation starting from the explicit form of the *L*-matrix (5*a*) and the Poisson brackets (9) and making use of a number of elliptic relations which are listed in the appendix. We will not give any details, but just restrict ourselves to giving a few intermediate relations, which can be established using the formulae from the appendix, namely

$$\begin{aligned}(L_\kappa\otimes L_{\kappa'})\,s_\kappa = \sum_{ij}\sum_{i'j'}&h_i^2h_{i'}^2\Phi_\kappa(q_i-q_j+\lambda)\Phi_{\kappa'}(q_{i'}-q_{j'}+\lambda)\,e_{ij}\otimes e_{i'j'}\\ &\times\delta_{jj'}\left[\zeta(\kappa+q_i-q_j+\lambda)-\zeta(q_i-q_j+\lambda)\right]\end{aligned} \tag{15a}$$

$$\begin{aligned}(L_\kappa\otimes\mathbf{1})\,s_\kappa\,(\mathbf{1}\otimes L_{\kappa'}) = \sum_{ij}\sum_{i'j'}&h_i^2h_{i'}^2\Phi_\kappa(q_i-q_j+\lambda)\Phi_{\kappa'}(q_{i'}-q_{j'}+\lambda)\,e_{ij}\otimes e_{i'j'}\\ &\times\delta_{ji'}\left[\zeta(\kappa+q_i-q_j+\lambda)-\zeta(q_i-q_j+\lambda)\right]\end{aligned} \tag{15b}$$

† In (13*c*) by the matrix $\partial_\lambda L_\kappa$ we mean

$$\partial_\lambda L_\kappa = \sum_{i,j=1}^{N} h_i^2\Phi_\kappa(q_i-q_j+\lambda)\left[\zeta(\kappa+q_i-q_j+\lambda)-\zeta(q_i-q_j+\lambda)\right]e_{ij}$$

i.e. we differentiate only with respect to the explicit dependence on the parameter λ.

as well as

$$\big[r^0_{\kappa,\kappa'}\,,\ L_\kappa\otimes L_{\kappa'}\big] = \sum_{ij}\sum_{i'j'} h_i^2 h_{i'}^2 \Phi_\kappa(q_i-q_j+\lambda)\Phi_{\kappa'}(q_{i'}-q_{j'}+\lambda)\, e_{ij}\otimes e_{i'j'}$$

$$\times\Big\{(1-\delta_{ii'})(1-\delta_{jj'})\big[\zeta(q_i-q_{i'})+\zeta(q_{i'}-q_j+\lambda)+\zeta(q_j-q_{j'})$$

$$-\zeta(q_i-q_{j'}+\lambda)\big]+\delta_{jj'}(1-\delta_{ii'})\big[\zeta(q_i-q_{i'})-\zeta(\kappa+q_i-q_j+\lambda)$$

$$+\zeta(\kappa'+q_{i'}-q_{j'}+\lambda)+\zeta(\kappa-\kappa')\big]+\delta_{ii'}(1-\delta_{jj'})\big[\zeta(q_j-q_{j'})$$

$$+\zeta(\kappa+q_i-q_j+\lambda)-\zeta(\kappa'+q_{i'}-q_{j'}+\lambda)-\zeta(\kappa-\kappa')\big]\Big\}\ . \qquad (15c)$$

We remark here that our r-matrices do not depend on momenta, like in the non-relativistic case [2], which was the motivation for the choice of the gauge of L_κ.

As a direct application of the r-matrix structure let us calculate the (continuous) time part of the Lax representation. It is obtained from the following formula:

$$(\mathrm{tr}\otimes id)(L_\kappa\otimes\mathbf{1})(r^+_{\kappa,\kappa'}-s^+_{\kappa,\kappa'}) = \Phi_{\kappa-\kappa'}(\lambda)L_{\kappa'}-\Phi_\kappa(\lambda)N_{\kappa'} \qquad (16)$$

where

$$N_\kappa = \sum_i\left[\zeta(\kappa)h_i^2+\sum_{j\neq i}h_j^2\zeta(q_i-q_j)-\sum_j h_j^2\zeta(q_i-q_j-\lambda)\right]e_{ii}+\sum_{i\neq j}h_i^2\Phi_\kappa(q_i-q_j)e_{ij} \qquad (17)$$

which, together with (5*a*), leads to the Lax representation found in [16] for the continuous RS model (up to a gauge transformation!). Thus, the continuous equations of motion (1*a*) corresponding to the Hamiltonian tr L_κ follow from the Lax equation

$$\dot{L}_\kappa = [N_\kappa\,,\ L_\kappa]\ . \qquad (18)$$

Remarks. (i) The non-relativistic limit is obtained by letting $\lambda\to 0$ while scaling the momenta $p_i := \lambda p_i$ and making the canonical transformation $p_i := p_i+\sum_{k\neq i}\zeta(q_i-q_k)$ such that $h_i^2\to 1+\lambda p_i+\mathrm{O}(\lambda^2)$ in (8). The r-matrix structure is linear in that limit since the L-matrix behaves as

$$L_\kappa\to\lambda^{-1}+\zeta(\kappa)+\sum_i p_i e_{ii}+\sum_{i\neq j}\Phi_\kappa(q_i-q_j)e_{ij}+\mathrm{O}(\lambda)$$

whereas the matrices $r^\pm_{\kappa,\kappa'}$, $s^\pm_{\kappa,\kappa'}$ enter in the following combination:

$$r^+_{\kappa,\kappa'}+s^-_{\kappa,\kappa'}\to r^{(\mathrm{nr})}_{\kappa,\kappa'}+\mathrm{O}(\lambda) \qquad (19)$$

in which the non-relativistic r-matrix is given by

$$r^{(\mathrm{nr})}_{\kappa,\kappa'} = \sum_i\big(\zeta(\kappa-\kappa')+\zeta(\kappa')\big)\,e_{ii}\otimes e_{ii}+\sum_{i\neq j}\Phi_{\kappa-\kappa'}(q_i-q_j)e_{ij}\otimes e_{ji}$$

$$+\sum_{i\neq j}\Phi_{\kappa'}(q_i-q_j)e_{jj}\otimes e_{ij} \qquad (20)$$

thus recovering the result of [2] in the leading terms.

(ii) We do not write down any Yang–Baxter-type relations between the r-matrices given in (12), because as a consequence of their dynamical nature the Yang–Baxter algebra does not seem to be closed, i.e. the Yang–Baxter 2-cocycle consists of terms which contain Poisson brackets with the L-matrix itself. It is an interesting open problem to see whether one can close the algebra on any level, leading to a possible truncation of some higher-order Yang–Baxter cocycle. So far, no results along this direction exist.

In this letter we have presented a classical r-matrix structure of the full elliptic Ruijsenaars–Schneider model. Since the model is the most general among the Calogero–Moser type models for the sl_n Lie algebra, our result is in a sense conclusive. Nonetheless, a number of questions have yet to be answered. Since the dynamical nature of the r-matrices implies that the corresponding Yang–Baxter algebra is not closed, it is not yet clear how to use it for quantization.

Concerning the quantization problem, the recent result of Hasegawa [20] should be mentioned, who has found an interesting connection between the quantum L-operator associated with Belavin's R-matrix and the quantum integrals of the Ruijsenaars' model. This somehow implies that on the classical level there should exist a gauge transformation involving the dynamical variables between an elliptic r-matrix of Belavin type and the one we have constructed in this letter.

Another result concerns the separation of variables approach leading to the explicit integral representations for the Macdonald polynomials associated with the trigonometric RS model, see [21]. So far the only result for the elliptic case is the separation of variables for the three-particle non-relativistic Calogero–Moser model [22]. The elliptic r-matrix could presumably help in constructing a separation of variables in the general case.

One more possible application of the results of this letter could lie in the time discretization of the RS model constructed in [14]. One feature of the proposed time discretization is that these discrete models share the time-independent part of the Lax pair with the corresponding continuous models and, consequently, the invariants take the same form in both cases. Thus, the proof of Liouville integrability (or the involutivity of the invariants) is exactly the same as for the continuous model. For the discrete models it seems that the most prominent role is played by the M-matrix. In fact, in similar systems related to integrable lattices it was found that there exists an extended Yang–Baxter structure which incorporates the M-matrix as well as the L-matrix in the Yang–Baxter algebra. Now that a classical r-matrix structure is available for the generic RS system, it would also be interesting to search for similar extended YB structures for these many-body systems.

VBK is supported by EPSRC under Grant No GR/K63887. OR is grateful to the Department of Applied Mathematical Studies of the University of Leeds for its hospitality during his visit.

Appendix

Here, we collect some useful formulas for elliptic functions, see also the standard textbooks, e.g. [23]. The Weierstrass sigma-function is defined by

$$\sigma(x) = x \prod_{(k,\ell)\neq(0,0)} \left(1 - \frac{x}{\omega_{k\ell}}\right) \exp\left[\frac{x}{\omega_{k\ell}} + \frac{1}{2}\left(\frac{x}{\omega_{k\ell}}\right)^2\right] \tag{A1}$$

with $\omega_{kl} = 2k\omega_1 + 2\ell\omega_2$ and $2\omega_{1,2}$ being a fixed pair of the primitive periods. The relations between the Weierstrass elliptic functions are given by

$$\zeta(x) = \frac{\sigma'(x)}{\sigma(x)} \qquad \wp(x) = -\zeta'(x) \tag{A2}$$

where $\sigma(x)$ and $\zeta(x)$ are odd functions and $\wp(x)$ is an even function of its argument. From an algebraic point of view, the most important property of these elliptic functions is the existence of a number of functional relations, the most fundamental being

$$\Phi_\kappa(x)\Phi_\kappa(y) = \Phi_\kappa(x+y)\left[\zeta(\kappa) + \zeta(x) + \zeta(y) - \zeta(\kappa + x + y)\right] \tag{A3}$$

The famous three-term relation for $\sigma(x)$ can be cast in the following convenient form:

$$\Phi_\kappa(x)\Phi_\lambda(y) = \Phi_\kappa(x-y)\Phi_{\kappa+\lambda}(y) + \Phi_{\kappa+\lambda}(x)\Phi_\lambda(y-x) \tag{A4}$$

which is obtained from the elliptic analogue of the partial fraction expansion, i.e. equation (A3).

There are a few additional important identities that are used in the proof of the r-matrix structure, the main one being given by

$$\begin{aligned}\Phi_{\kappa-\kappa'}(a-b)\Phi_\kappa(x+b)\Phi_{\kappa'}(a+y) - \Phi_{\kappa-\kappa'}(x-y)\Phi_\kappa(y+a)\Phi_{\kappa'}(x+b)\\ = \Phi_\kappa(x+a)\Phi_{\kappa'}(y+b)\left[\zeta(a-b)+\zeta(x+b)-\zeta(x-y)-\zeta(y+a)\right]\end{aligned} \tag{A5}$$

which can be derived from (A4), together with (A3), and

$$\begin{aligned}\Phi_{\kappa-\kappa'}(x-y)\Phi_\kappa(y+a)\Phi_{\kappa'}(x+a)\\ = \Phi_\kappa(x+a)\Phi_{\kappa'}(y+a)\left[\zeta(x-y)-\zeta(\kappa+x+a)+\zeta(\kappa'+y+a)\right.\\ \left.+\zeta(\kappa-\kappa')\right].\end{aligned} \tag{A6}$$

It is equations (A5) and (A6) that are used in the derivation of (15*c*) which forms the main step in the computation of the r-matrix.

In [14] an elliptic version of the Lagrange interpolation formula was used, which was derived on the basis of an elliptic version of the Cauchy identity. We can write the elliptic Cauchy identity in the following elegant form:

$$\det\left(\Phi_\kappa(x_i-y_j)\right) = \Phi_\kappa(\Sigma)\sigma(\Sigma)\frac{\prod_{k<\ell}\sigma(x_k-x_\ell)\sigma(y_\ell-y_k)}{\prod_{k,\ell}\sigma(x_k-y_\ell)}$$

$$\text{where } \Sigma \equiv \sum_i (x_i-y_i)\,. \tag{A7}$$

An elliptic form of the Lagrange interpolation formula is obtained by expanding (A7) along one of its rows or columns. Thus, we obtain

$$\prod_{i=1}^N \frac{\sigma(\xi-x_i)}{\sigma(\xi-y_i)} = \sum_{i=1}^N \Phi_{-\Sigma}(\xi-y_i)\frac{\prod_{j=1}^N\sigma(y_i-x_j)}{\prod_{j=1, j\neq i}^N \sigma(y_i-y_j)} \qquad \text{when } \Sigma = \sum_{i=1}^N (x_i-y_i)\neq 0 \tag{A8}$$

and

$$\prod_{i=1}^N \frac{\sigma(\xi-x_i)}{\sigma(\xi-y_i)} = \sum_{i=1}^N \left[\zeta(\xi-y_i)-\zeta(x-y_i)\right]\frac{\prod_{j=1}^N\sigma(y_i-x_j)}{\prod_{j=1, j\neq i}^N \sigma(y_i-y_j)}$$

$$\text{when } \sum_{i=1}^N (y_i-x_i) = 0 \tag{A9}$$

(here x denotes one of the zeros x_i). Note that in this case the left-hand side is a meromorphic function on the elliptic curve as a consequence of Abel's theorem. It can be easily verified that equation (A9) is independent of the choice of x as a consequence of the relation

$$\sum_{i=1}^N \frac{\prod_{j=1}^N\sigma(y_i-x_j)}{\prod_{j=1, j\neq i}^N \sigma(y_i-y_j)} = 0 \qquad \text{when } \sum_{i=1}^N (y_i-x_i) = 0 \tag{A10}$$

(see, e.g., [23, p 451]), which follows from equation (A8) in the limit $\Sigma\to 0$.

Finally, we give the expression for the inverse of the elliptic Cauchy matrix, namely

$$\left[(\Phi_\kappa(x_\cdot - y_\cdot))^{-1}\right]_{ij} = \Phi_{\kappa+\Sigma}(y_i - x_j)\frac{P(y_i)Q(x_j)}{Q_1(y_i)P_1(x_j)} \tag{A11}$$

(with Σ as before), in terms of the elliptic polynomials

$$P(\xi) = \prod_{k=1}^{N}\sigma(\xi - x_k) \qquad Q(\xi) = \prod_{k=1}^{N}\sigma(\xi - y_k)$$

and

$$P_1(x_j) = \prod_{k\neq j}\sigma(x_j - x_k) \qquad Q_1(y_i) = \prod_{k\neq i}\sigma(y_i - y_k)\,. \tag{A12}$$

Equation (A11) can be derived using (A8) and (A9), and is used to derive equation (14) in the main text.

References

[1] Avan J and Talon M 1993 Classical *R*-matrix structure for the Calogero model *Phys. Lett.* **303B** 33–7

[2] Sklyanin E K 1994 Dynamical *r*-matrices for elliptic Calogero–Moser model *St. Petersburg J. Math.* **6** 227

[3] Braden H W and Suzuki T 1994 *R*-matrices for the elliptic Calogero–Moser models *Lett. Math. Phys.* **30** 147

[4] Eilbeck J C, Enol'skii V Z, Kuznetsov V B and Tsiganov A V 1994 Linear *r*-matrix algebra for classical separable systems *J. Phys. A: Math. Gen.* **27** 567

[5] Avan J, Babelon O and Billey E 1995 The Gervais–Neveu–Felder equation and the quantum Calogero–Moser systems *Preprint* PAR LPTHE 95-25; *Preprint* hep-th/9505091 (*Commun. Math. Phys.* to be published)

[6] Ruijsenaars S N M and Schneider H 1986 A new class of integrable systems and its relation to solitons *Ann. Phys.* **170** 370–405

[7] Ruijsenaars S N M 1987 Complete integrability of relativistic Calogero–Moser systems and elliptic function identities *Commun. Math. Phys.* **110** 191–213

[8] van Diejen J-F 1995 Commuting difference operators with polynomial eigenfunctions *Compos. Math.* **95** 183–233

[9] Macdonald I G 1995 *Symmetric Functions and Hall Polynomials* 2nd edn (Oxford: Oxford University Press)

[10] Avan J and Rollet G 1996 The classical *r*-matrix for the relativistic Ruijsenaars–Schneider system *Phys. Lett.* **212A** 50–4

[11] Babelon O and Bernard D 1993 The sine-Gordon solitons as a *N*-body problem *Phys. Lett.* **317B** 363–8

[12] Suris Yu B 1996 Why are the Ruijsenaars–Schneider and the Calogero–Moser hierarchies governed by the same *r*-matrix? *University of Bremen preprint* hep-th/9602160

[13] Krichever I and Zabrodin A 1995 Spin generalization of the Ruijsenaars–Schneider model, non-Abelian 2D Toda chain and representations of Sklyanin algebra *Landau Institute preprint* hep-th/9505039

[14] Nijhoff F W, Ragnisco O and Kuznetsov V B 1996 Integrable time-discretization of the Ruijsenaars–Schneider model *Commun. Math. Phys.* **176** 681–700

[15] Veselov A P 1992 Growth and integrability in the dynamics of maps *Commun. Math. Phys.* **145** 181–93

[16] Bruschi M and Calogero F 1987 The Lax representation for an integrable class of relativistic dynamical systems *Commun. Math. Phys.* **109** 481–92

[17] Maillet J-M 1985 Kac–Moody algebra and extended Yang–Baxter relations in the $O(N)$ non-linear σ-model *Phys. Lett.* **162B** 137–42

[18] Freidel L and Maillet J-M 1991 Quadratic algebras and integrable systems *Phys. Lett.* **262B** 278–84

[19] Nijhoff F W, Papageorgiou V G and Capel H W 1992 *Quantum Groups (Springer Lecture Notes in Mathematics 1510)* ed P P Kulish (Berlin: Springer) p 312

[20] Hasegawa K 1995 Ruijsenaars' commuting difference operators as commuting transfer matrices *Tohoku University preprint* q-alg/9512029

[21] Kuznetsov V B and Sklyanin E K 1996 Separation of variables for A_2 Ruijsenaars model and new integral representation for A_2 Macdonald polynomials *Preprint* q-alg/9602023 (*J. Phys. A: Math. Gen.* submitted)

[22] Sklyanin E K 1995 Separation of variables. New trends *Progr. Theor. Phys. Suppl.* **118** 35–60

[23] Whittaker E T and Watson G N 1988 *A Course in Modern Analysis* 4th edn (Cambridge: Cambridge University Press)